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General Relativistic Theory of Gravitation Exhibiting Complete Group Covariance

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The geometry of the group manifold is generalized to allow the construction of a general relativistic theory exhibiting all the degrees of freedom of the group in its dynamical variables. The theory has features of a multidimensional Kaluza–Klein theory with noncommuting Killing vectors. The resulting fields suggest, however, a teleparallelistic formulation. The geometry and gauge invariance of the theory are discussed.

1. INTRODUCTION

The mathematical problem of de Sitter covariant field equations was treated early by Dirac (Dirac, 1935). The first suggestion to construct a de Sitter covariant general relativistic theory was probably made by Lubkin (Lubkin, 1971). A formulation based on a generalization of Pauli's version of the principle of equivalence was performed by the present author (Halpern, 1977).

A large number of publications have appeared since on de Sitter covariant general relativity. A common feature of the vast majority of publications on the subject seems to be that de Sitter covariance is assumed at the beginning when dealing with the simplest situations. Later the large radius of the universe is claimed to make the transition to a Poincaré invariant theory. The introduction of de Sitter covariance appears then rather as an artifice to achieve elegant formulations of Poincaré covariant theories.

There is, however, a fundamental difference between the simple de Sitter (and anti-de Sitter) group and the Poincaré group, and if de Sitter covariance would indeed be a better approximation for the description of nature (for which we certainly do not have any evidence yet), then we may expect deep-going consequences for physics.

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The spirit from which the present work originated is to assume the covariance as fundamentally true although broken by perturbing conditions. The motive is rather the beauty of the symmetry than a Machian justification. The conclusion is that any possible degree of freedom that can be associated with the symmetry should be considered.

We start out in this spirit from an arbitrary group manifold and consider all the degrees of freedom of the group as candidates for dynamical variables.

A method to arrive from the space-time manifold to the group manifold has been briefly presented (Halpern, 1979).

The method consists in extension of the representation space of the group of transformations starting from space-time and including the transformation of the first and successively higher differentials until the orbit of the group in the resulting higher-dimensional space forms the group manifold.

Section 2 of the present work recalls some basic relations of the group manifold and its geometry. We are mainly interested in local properties. The relation of the geometry of the group space to that of space-time is demonstrated.

These geometrical properties and the previous considerations suggest a generalization of the group manifold to the space of a multidimensional theory of the Kaluza–Klein type (Kaluza, 1921). The noncommutativity of the Killing vectors requires, however, generalizations and modifications. The mathematical aspects are presented in Sections 3 and 4.

The formalism stimulates many physical questions; it predicts a cosmological inertial field of the electromagnetic type and additional degrees of freedom. The form of the fields suggests a teleparallelistic generalization of the theory. Right when parity violation was discovered Pauli and many other physicists suspected that it may be caused by the existence of higher-dimensional curved spaces. The present formulation seems to indicate a somewhat different geometric relation associated with noncommutativity of the generators.

A second publication on the subject which is now in preparation should discuss the questions arising in the present article in full detail.

2. REVIEW OF THE GEOMETRY OF THE GROUP MANIFOLD OF A SEMISIMPLE GROUP

A detailed presentation of the geometry of the group manifold is given in Eisenhart (1933) and a shorter review for physicists in De Witt (1963). We only remind the reader here of some fundamental relations of importance to us and fix the notation. The proofs for theorems which we shall need are given in the following sections, in a form which allows them to be applied to generalizations of the group manifold.

We denote a basis of the tangent vectors on the group manifold which generate left-invariant infinitesimal right translations by A_R ($R=1\cdots r$; the dimension of the group space); a corresponding basis of right-invariant infinitesimal left translations is denoted by $\bar{A}_R$. The former form a basis of the Lie algebra of the group, the latter of that of the reciprocal group (Eisenhart, 1933):

$$[A_R, A_S]=C^T_{RS}A_T, \qquad [\bar{A}_R, \bar{A}_S]=C^T_{SR}\bar{A}_T$$

$$[A_R, \bar{A}_S]=0 \tag{1}$$

(Basis indices are capital Latin letters; coordinate indices are lower case Latin letters.)

The complementary left- and right-invariant forms are denoted by A^R and $\bar{A}^R$, respectively. Given a law of composition of the group:

$$z^r=\Phi^r(x,y) \qquad \text{or short: } z=xy \tag{2}$$

we have in components

$$A^R_r=\left[\frac{\partial\Phi^R(x^{-1},y)}{\partial y^r}\right]_{y=x}, \qquad \bar{A}^R_r=\left[\frac{\partial\Phi^R(y,x^{-1})}{\partial y^r}\right]_{y=x}, \tag{2a}$$

and, e.g., the left invariance of A^R resulting from the group property states

$$A^R_r(z)\frac{\partial\Phi^r(x,y)}{\partial y^s}=A^R_s(y) \tag{2b}$$

The exterior derivatives fulfill

$$dA^R+C^R_{ST}A^SA^T=0 \tag{1a}$$

$$d\bar{A}^R-C^R_{ST}\bar{A}^S\bar{A}^T=0 \tag{1b}$$

A semisimple group has a nonsingular metric:

$$\gamma_{RS}=C^U_{RV}C^V_{US} \tag{3}$$

which can be diagonalized with values ± 1 by a linear transformation of the base. A Riemannian metric is obtained for the group space:

$$g_{uv} = A_u^R \gamma_{RS} A_v^S = \bar{A}_u^R \gamma_{RS} \bar{A}_v^S \tag{3a}$$

At every point A^R, $\bar{A}^R$ are related by an adjoint transformation which does not alter the structure constants.

Because of equations (1) the metric obeys Killing's equations:

$$g_{uv,t} A^t + g_{tv} A^t{}_{,u} + g_{ut} A^t{}_{,v} = 0 \tag{4}$$

for all A which are linear superpositions of the A_R and $\bar{A}_S$. A_R as well as $\bar{A}_R$ form orthogonal n-tuples in the group manifold with metric (3a). The metric (3a) is that of an Einstein space with Ricci tensor:

$$R^t_{uvt} = R_{uv} = -\tfrac{1}{4} g_{uv} \tag{5}$$

Subgroups of the reciprocal group have orbits which form systems of imprimitivity in the group manifold (Eisenhart, 1933).

A coordinate system can in general be introduced in which the matrix of the components $(\bar{A}_R^s)$ has the following form:

$$(\bar{A}_R^s) = \begin{pmatrix} \bar{A}_J^i(x^s), & 0 \\ \bar{A}_J^m(x^s), & \bar{A}_M^m(x^n) \end{pmatrix} \tag{6}$$

Here coordinates labeled by m, n and basis indices M belong to the subgroup and its orbits in the group manifold, whereas coordinates and basis indices labeled i, j and I, J belong to the complementary sets and r, s and R, S run over all indices.

For example the de Sitter and anti-de Sitter groups $O(4,1)$ and $O(3,2)$ have each ten parameters and six-parameter subgroups. The de Sitter space time is the factor space with the contravariant metric:

$$g^{ik}(x) = \bar{A}_R^i \gamma^{RS} \bar{A}_S^k \tag{3'}$$

due to the particular form of (6) and equations (1) $g^{ik}(x^i)$ does not depend on x^m $(m>4)$ although $\bar{A}_J^i$ does. x^i can then be considered as coordinates of space-time. The covariant metric of space-time is not that of equation (3a) but rather the algebraic complement of g^{ik} $(i, k \leq 4)$ above. We shall see that the mixed components g_{im} can in this case not all be made to vanish everywhere by a coordinate transformation.

The trajectories of the group are geodesics on the group manifold with the metric (3a).

3. GENERALIZATION OF THE FORMALISM

The de Sitter space time emerges in Section 2 from the factor space of the ten-parameter group with respect to a six-parameter subgroup of the reciprocal group. The metric on the group manifold has been obtained by choosing the basis vectors as an orthogonal n-tuple (decuple in the de Sitter case). The metric of de Sitter space-time is then obtained by choosing the basis vectors of the factor space as an orthogonal quadruple (tetrades) [see equations (6) of Section 2]. The metric of space-time does not depend on the coordinates x^m of the subgroup; however, that of the tetrads obtained from the basis vectors of the reciprocal group does. This dependence is, however, only on adjoint transformations of the first four basis vectors among themselves (tetrad rotations) and seems not physically significant.

Application of the method up till now is of course not limited to the de Sitter group. Any semisimple group with suitable subgroup can serve the purpose.

A main idea of the present work is that if the semisimple invariance group is truly fundamental, then all its operations should be as far as possible on an equal footing. The fact that also the nongeodesic timelike group trajectories in de Sitter space allow a physical interpretation (for example as due to charges in special electromagnetic fields) serves as an encouragement for this point of view (Halpern 1980).

The de Sitter space corresponds to the absence of localized matter distribution. We have to generalize it to describe physics. The procedure adopted here is to leave the six-dimensional subgroup and the space of the tetrads as factor space. Thus equation (6) still looks the same, but the first four members of the (still orthogonal) decuple (we denote them here by $\bar{B}_F$ instead of $\bar{A}_F$) do not belong in general to a Lie algebra. The space-time metric formed as before out of the decuple, still does not depend on x^m.

A minimum requirement which must be imposed, however, is that the last six $\bar{B}_M(x^n)=\bar{A}_M$ remain Killing vectors; otherwise the remaining six dimensions would become observable. We adopt thus a main point of view of the Kaluza–Klein theory. (Kaluza, 1928; Einstein, 1927; Klein). The difference from other multidimensional generalizations of this theory is that the most symmetric case is that of the manifold of a non-Abelian (here even simple) group and that the present theory (or at least some versions of it) is even teleparallelistic. Motions exist also in non-Riemannian spaces. The generalized Riemannian metric or the $\bar{B}_R$ must be determined by field equations in ten dimensions which will have, in general, a source term. To see what the condition that $\bar{B}_M = \bar{A}_M(x^m)$ remain Killing vectors implies we write the complementary matrix to the $\bar{B}_R$ in the same coordinate system:

$$(\bar{B}_s^R) = \begin{pmatrix} \bar{B}_i^J(x^t), & \bar{B}_i^M(x^t) \\ 0, & B_m^M(x^n) \end{pmatrix} \tag{6a}$$

We have

$$\bar{B}_t^R \bar{B}_S^t = \delta_s^R, \qquad \bar{B}_t^R \bar{B}_R^r = \delta_t^r \tag{6b}$$

but also

$$\bar{B}_k^J \bar{B}_H^k = \delta_H^J, \qquad \bar{B}_i^J \bar{B}_J^k = \delta_i^k \tag{6b$'$}$$

where according to our rule the indices in the second set of equations run from 1 to 4. The metric is accordingly

$$g_{ik} = \bar{B}_i^E \gamma_{EF} \bar{B}_k^F + \bar{B}_i^M \gamma_{MN} \bar{B}_k^N \tag{3b}$$

$$g_{im} = \bar{B}_i^M \gamma_{MN} \bar{B}_m^N, \qquad g_{mn} = \bar{B}_m^M \gamma_{MN} \bar{B}_n^N \tag{3c}$$

Killing's equations for vectors $\bar{B}_M$ require in this case

$$g_{ik,m} = 0 \tag{3d}$$

$$g_{im,n} \bar{B}_M^n + g_{in} \bar{B}_{M,m}^n = 0 \tag{3e}$$

The first part of g_{ik}, equation (3b), is the metric of space-time which is independent of x^m; the second part as well as the g_{im} fulfill Killing's equations iff:

$$\bar{B}_{i,m}^M = -c_{PQ}^M \bar{B}_i^P \bar{B}_m^Q \tag{3f}$$

Killing's equations for g_{mn} are fulfilled because

$$\bar{B}_M^t = \bar{A}_M^t, \qquad \bar{B}_m^M = \bar{A}_m^M$$

Next we show that even for the de Sitter case coordinate transformations

$$x'^i = x^i, \qquad x'^m = x'^m(x^k, x^n) \tag{7}$$

which are the analog of gauge transformations in the Kaluza–Klein theory cannot remove all the g^{im} and consequently also not the g_{im}:

Proof. $g^{im} \doteq 0$ implies for (7)

$$g'^{is}\frac{\partial x'^m}{\partial x^s}=0 \quad \text{and} \quad \bar{A}^s_E\frac{\partial x'^m}{\partial x^s}=0 \tag{7a}$$

In the de Sitter case, however,

$$\left[\bar{A}_E, \bar{A}_F\right]=c^M_{EF}\bar{A}_M \tag{8}$$

so that also the six equations

$$\bar{A}^n_M\frac{\partial x'^m}{\partial x^n}=0 \tag{7b}$$

must be fulfilled, which is not possible except for constants x'^m.

This shows us that the g_{im} are not suitable potentials for the present generalizations of the Kaluza theory.

We find that the $\bar{B}^R$ are suitable potentials and obtain thus a theory with teleparallelism.

The fields in this theory are then

$$F^R=d\bar{B}^R-c^R_{ST}\bar{B}^S\bar{B}^T \tag{1c}$$

The requirement that $\bar{B}_M=\bar{A}_m$ be Killing vectors translates then into

$$F^M_{in}=0, \qquad F^M_{mn}=0 \tag{1d}$$

If we impose in addition

$$F^E_{im}=0, \qquad F^E_{mn}=0 \tag{1e}$$

we obtain

$$\bar{B}^E_{i,m}=-c^E_{FM}\bar{B}^M_m\bar{B}^F_i, \qquad \bar{B}^i_E=c^F_{EM}\bar{B}^M_m\bar{B}^i_E \tag{9}$$

and

$$F^R_{ik,m}=-c^R_{SM}\bar{B}^M_m F^S_{ik} \tag{9a}$$

which says that displacement in the direction of the Killing vectors $\bar{B}_M$ generates only an adjoint transformation of the fields F^R_{ik} so that the tensor components

$$F_{i\cdot k}^{\ j} = F_{ik}^{R} \bar{B}_R^i \tag{10}$$

are independent of such displacements and depend thus only on the space-time coordinates.

The case where in addition to equations (9) also the tensor

$$F_{rst} = F_{[r,s,t]} \tag{11}$$

is completely skew symmetric shows the greatest regularity while still allowing a more general metric of space-time:

$$F_{rst} = 0 \qquad \text{if } r, s, \text{ or } t > 4$$

because also $F_{mi}^R = 0$. This requirement is equivalent to

$$F^M = 0 \tag{11a}$$

as one can easily see.

Theorem. If

$$F_{uvw} = \left(\bar{B}^R_{w,u} - \bar{B}^R_{u,w} + C^R_{ST}\bar{B}^S_w\bar{B}^T_u\right)\gamma_{RY}\bar{B}_v^{\ Y} \tag{12}$$

is totally skew symmetric, then all the $\bar{B}^R$ are Killing vectors and all their trajectories are geodesics.

Proof. $C_{SYT} = \gamma_{RY}C^R_{ST}$ is totally skew symmetric and thus also $C_{SYT}\bar{B}^S_w\bar{B}^T_u\bar{B}^Y_v$ and consequently also the remaining right-hand term in equation (12). This term can be written as

$$(\gamma_{YST} - \gamma_{YTS})\bar{B}_v^{\ Y}\bar{B}^S_w\bar{B}_u^{\ T} \text{ with the Ricci rotation coefficients:}$$

$$\gamma_{YST} = \bar{B}^R_{w;u}\bar{B}^w_S\bar{B}^u_T\gamma_{RY} \tag{12a}$$

because of antisymmetry for $Y = S$: $\gamma_{TSS} = 0$, which is the condition that the trajectory of $\bar{B}_T$ is a geodesic (Eisenhart, 1961).

Furthermore with the metric of the orthogonal n-tuple Killing equations for a vector $\bar{B}_E$ are

$$\gamma_{UV}\left(B^U_{r,t}\bar{B}^t_E\bar{B}_s^{\ V} + \bar{B}^U_r\bar{B}^{\ V}_{s,t}\bar{B}^t_E + \bar{B}^U_r\bar{B}_t^{\ V}\bar{B}^t_{E,s} + B^U_t\bar{B}_s^{\ V}\bar{B}^t_{E,r}\right)$$

$$= \gamma_{UV}\left[\left(\bar{B}^U_{s,t} - \bar{B}^U_{t,r}\right)\bar{B}_s^{\ V}\bar{B}^t_E + \left(\bar{B}^V_{s,t} - \bar{B}^V_{t,s}\right)\bar{B}^U_r\bar{B}^t_E\right] = F_{rst} + F_{srt} + C_{tsr} + C_{trs} = 0 \tag{13}$$

because of the antisymmetry. Thus every $\bar{B}_E$ is also a Killing vector. The space-time metric need not be just that of the group space in order not to be interfering with these properties.

The following section compares geodesics in the ten-dimensional space and in space-time.

4. GEODESICS AND THEIR PROJECTION ON SPACE-TIME

According to the theorem of the last section, the trajectory of every $\bar{B}_R$ is a geodesic if F_{rst} is skew symmetric.

$\bar{B}_R$ are unit vectors and we can introduce the parameter Υ of a given geodesic such that $\dot{x}^r = \bar{B}^r_R C^R = B^r$ with constant C^R. On the points of the geodesic we have also

$$\ddot{x}^r = B^r_{\ ,t} B^t \tag{14}$$

Let us introduce the coordinate system used in the previous sections where x^i $(i \leqslant 4)$ were also coordinates of space-time. We form now

$$\ddot{x}^i + \{j^i k\}\dot{x}^j \dot{x}^h = B^i_{\ ,t} B^t + \{j^i k\} B^j B^k = B^i_{\ ,k} B^k + B^i_{\ ,m} B^m + \{j^i k\} B^j B^k \tag{14a}$$

The Christoffel connection is here formed with the metric of space-time; expressing it by the space-time tetrads $\bar{B}^i_F$ one obtains from (14a)

$$B^i_{\ ,m} B^m + g^{il}\left(\bar{B}^G_{l,j} - \bar{B}^G_{j,l}\right)\gamma_{GB} C^B \bar{B}^j_A C^A \qquad (i, e, j, A, B, G \leqslant 4, \qquad m > 4) \tag{14b}$$

all the other terms cancel. In the special case considered previously, where

$$\bar{B}^i_{E,m} = C^{\ F}_{EM} \bar{B}^M_m \bar{B}^i_F \quad \text{and} \quad F^M = 0 \qquad \text{(antisymmetry)}$$

this equals

$$C^E C^{\ F}_{EM} \bar{B}^M_m \bar{B}^i_F \left(\bar{B}^m_H C^H + \bar{B}^m_M C^M\right) + g^{il} F_{jkl} \bar{B}^k_B C^B \bar{B}^j_A C^A + g^{il} C_{HBM}\left(\bar{B}^H_l \bar{B}^M_j\right) C^B \bar{B}^j_A C^A = F^i_k B^k \tag{14c}$$

with

$$F^i_{\ k} = C^{\ F}_{GM} C^M \bar{B}^i_F \bar{B}^G_k \tag{14d}$$

We see from equations (14a)–(14d) the following.

Theorem. the projection of a geodesic on space-time is a geodesic if $c^M=0$ ($M>4$), whereas it is equivalent to the trajectory of a charge in an electromagnetic field F^i_k if $c^M\neq 0$ (Halpern, 1980).

Conversely: Every geodesic of space-time is the projection of a geodesic in the r-dimensional space. If F_{jkl} is not antisymmetric (if $F^M\neq 0$), there occurs an additional term:

$$F^{i\,M}_{\;j}\gamma_{MN}\dot{x}^j \tag{14e}$$

Only $c^M\neq 0$ ($M>4$) results in nongeodesic motion.

5. PROBLEMS OF INVARIANCE

We have seen in the last section that totally antisymmetric fields result in a metric space of remarkable symmetry without impairing the generality of the space-time metric. With the restrictions of the x^m dependence which were imposed, antisymmetry is equivalent to $F^M=0$ and this case is no doubt closest to the case of general relativity with no other than gravitational fields present.

One can now impose a metric Lagrangian: following Kaluza the curvature invariant R plus a cosmological member to obtain the correct curvature of space in the absence of sources.

The Riemann tensor in group space can in case of a semisimple group be expressed by the structure constants and the $\bar{A}^R$:

$$R_{rstu}=\tfrac{1}{4}C_R{}^V{}_S C_{VTU}\bar{A}_r{}^R\bar{A}_s^S\bar{A}_t{}^T\bar{A}_u{}^U \tag{15}$$

and

$$R_{st}=-\tfrac{1}{4}g_{st},\qquad R_{st}-\tfrac{1}{2}g_{st}R=g_{st} \tag{15a}$$

so that in our units this cosmological member equals minus unity. Within laboratory dimensions we assume the cosmological member unobservably small. We assume our unit of length thus of the order of the radius of the universe.

Solutions of the field equations have to be restricted so that $\bar{B}_M=\bar{A}_M$ are Killing vectors.

The second derivatives of our field equations with respect to x^k are then the same as in Einstein's equations. As long as the derivatives of x^m are of cosmological smallness we obtain within the linear approximation order of magnitude agreement with general relativity.

The electromagnetic field in the Kaluza–Klein theory is expressed in terms of the metric tensor component g_{i5} as

$$F_{ik}=g_{k5,i}-g_{i5,k} \tag{16}$$

We have obtained in the previous sections more general expressions for ten fields F^R_{ik} which allow one even to express the absence of sources of the gravitational field. The formulation of field equations in terms of these fields is tempting.

The F^R have the well-known gauge covariance:

$$\begin{aligned} \bar{B}^R_t(x) &\to -\bar{B}^R_u(a(x))a^u(x)_{,t}+\rho^R_S(a(x))\bar{B}^S_t(x) \\ F^R &\to \rho^R_S(a(x))F^S \end{aligned}$$

with $\rho(a)$ the matrix of an adjoint transformation with group element a^r which may depend on the point x. (In group space $\bar{B}^R=\bar{A}^R$, $F^R=0$ and the right invariance implies also the invariance of $\bar{A}^R$). We prove the following.

> *Theorem.* If all $\bar{B}^R$ are Killing vectors the effect of the gauge transformation of equation (17) on the metric is equivalent to that of a coordinate transformation and a point transformation.

Proof. An infinitesimal transformation with

$$\rho^R_S(a)=\gamma^R_S+C^R_{ST}a^T$$

$$\bar{B}^R_t(x)\to -a^R(x)_{,t}+\bar{B}^R_t(x)+C^R_{ST}\bar{B}^S_t(x)a^T \tag{17a}$$

transforms the metric

$$g_{uv}=\gamma_{RS}\bar{B}^R_u\bar{B}^S_v\to g_{uv}-\gamma_{RS}\left(a^R_{,u}\delta^S_v+a^S_{,v}\delta^R_u\right)$$

$$=g_{uv}-g_{tv}\left(\bar{B}^t_Ta^T\right)_{,u}+g_{ut}\left(\bar{B}^t_Ta^T\right)_{,v}+\left(g_{tv}\bar{B}^t_{T,u}+g_{ut}\bar{B}^t_{T,v}\right)a^T \tag{17b}$$

because of Killings equations this equals

$$g'_{uv}(x)=g'_{uv}(x')-g_{uv,t}\bar{B}^t_Ta^T \tag{17c}$$

where $g'_{uv}(x')$ results from $g_{uv}(x)$ by an infinitesimal coordinate transformation:

$$x'=x+\bar{B}_Ta^T \tag{17d}$$

A Lagrangian formed out of the fields F^R and the metric has therefore to renounce on the beautiful gauge invariance [equation (17a)] because the metric which is formed out of the B^R transforms in general in a different way.

I think this is not a reason to exclude a theory with teleparallelism. The gauge invariance which exists in the source-free case of the group space seems not to cause difficulties because the $\bar{A}^R$ are not altered by the transformations.

Unless one succeeds in forming a Lagrangian of the fields which is free of the metric, which in ten dimensions would no doubt require bizarre nonlinear terms, best candidates are the ten-dimensional versions of the Lagrangians

$$\mathfrak{L}_{\mathrm{I}} = \sqrt{g}\, F^U_{rt} F^{rtV} \gamma_{UV} \tag{18a}$$

$$\mathfrak{L}_{\mathrm{II}} = \sqrt{g}\, F^U_{rs} \gamma_{UT} \bar{B}^T_t F^{rtV} \bar{B}^s_V \tag{18b}$$

which in four dimensions have already been considered in Einstein's teleparallelistic theory.

A more detailed analysis of these cases will follow in a subsequent publication.

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